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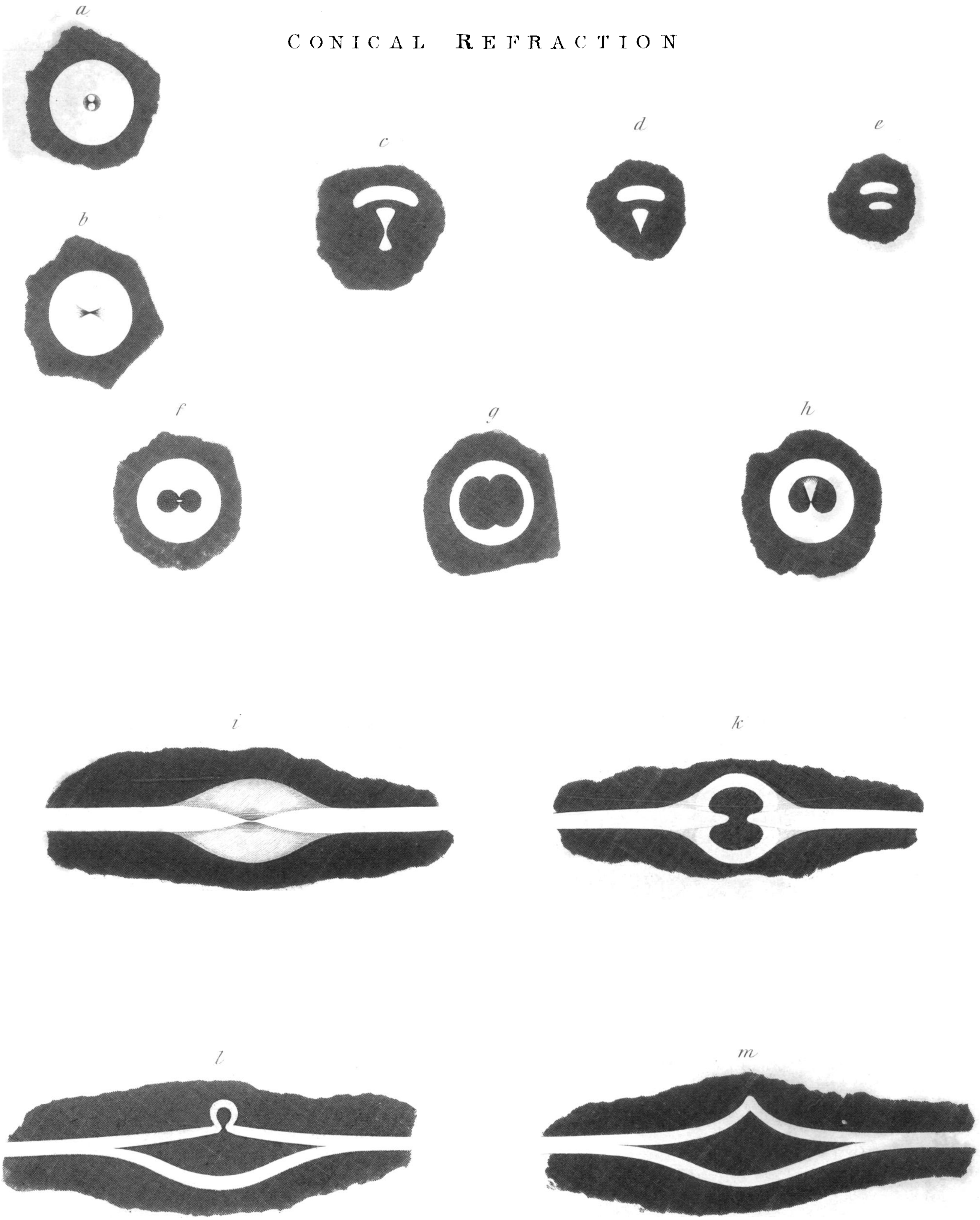
CONICAL REFRACTION



J. Kirkwood Sc Dublin.

CONICAL REFRACTION,

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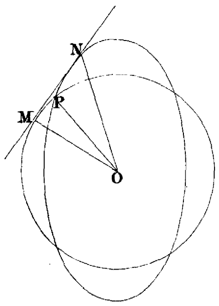

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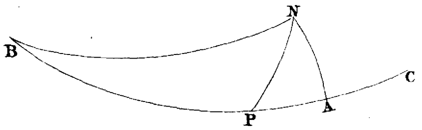

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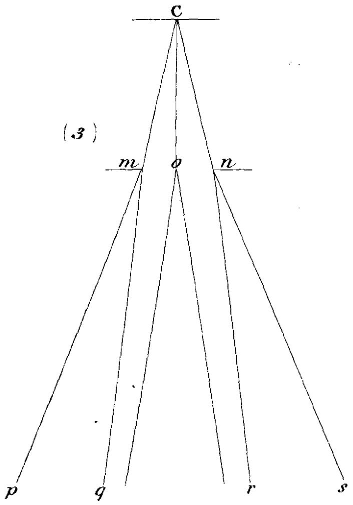

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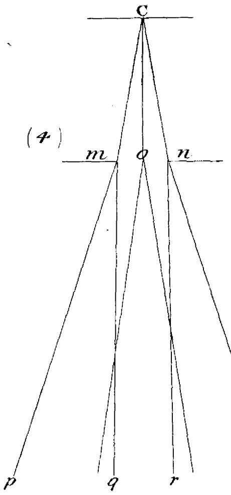

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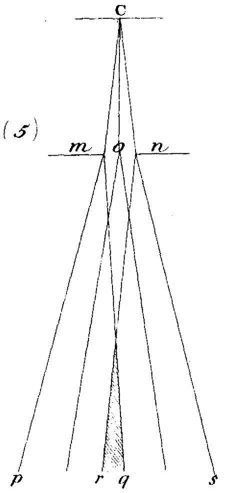

J. Kirkwood Sc. Dublin.

On the Phenomena presented by Light in its passage along the Axes of Biaxal Crystals. By the Rev. HUMPHREY LLOYD, A. M., M. R. I. A., *Fellow of Trinity College, and Professor of Natural and Experimental Philosophy in the University of Dublin.*

Read January 28, 1833.

It is well known that when a ray of light is incident upon certain crystals, such as Iceland spar and quartz, it is in general divided into two pencils, of which one is refracted according to the known law of the sines, while the direction of the other is determined by a new and extraordinary law, first assigned by Huyghens.

These laws were long supposed to apply to all doubly refracting substances; and it was not until the subject was examined by the ablest advocate of the undulatory theory, that the problem of double refraction was solved in all its generality. Setting out from the hypothesis, that the elasticity of the vibrating medium within the crystal is unequal in three rectangular directions, Fresnel has shown that the surface of the wave is not, in general, either a sphere or spheroid, as in the Huyghenian law, but a surface of the fourth order, consisting of two sheets; and that the directions of the two refracted rays are determined by tangent planes drawn to these surfaces under known conditions. Such crystals have, in general, two optic axes, and are thence denominated *biaxal*. When the elasticity of the medium is the same in two of the three directions, the equation of the wave-surface is resolvable into two, which represent the sphere and spheroid of the Huyghenian law. The two optic axes in this case coincide; and the law of Huyghens is thus proved to be a case of a more general law, and shown to belong to *uniaxal* crystals only. Finally, when the elasticity is the same in all the three directions, the wave-surface becomes a sphere; and the refraction is single, and takes place according to the ordinary law of the sines. This case comprises a few of the crystallized, and most uncrystallized substances.

There are two remarkable cases, however, in this elegant and profound theory, which its author seems to have overlooked, if not to have misapprehended. In a communication made to the Academy at its last meeting, Professor Hamilton has supplied these omissions in the theory of Fresnel, and has been thus led to results in the highest degree novel and remarkable.

To understand these conclusions, it may be useful to revert for a moment to the original theory of Fresnel. The general form of the wave-surface is determined by the equation

$$\begin{gathered}(a^2\cos.^2\alpha + b^2\cos.^2\beta + c^2\cos.^2\gamma)\,r^4 \\ -\left[a^2(b^2+c^2)\cos.^2\alpha + b^2(a^2+c^2)\cos.^2\beta + c^2(a^2+b^2)\cos.^2\gamma\right]r^2 \\ + a^2b^2c^2 = 0\,;\end{gathered}$$

in which α, β, γ, denote the angles made by the radius-vector with the three axes, and a^2, b^2, c^2, the elasticities of the medium in these directions. If now we make $\cos.\beta = 0$, in this equation, so as to obtain the section of the surface made by the plane of xz, the result is reducible to the form

$$(r^2 - b^2)\left[(a^2\cos.^2\alpha + c^2\sin.^2\alpha)\,r^2 - a^2c^2\right] = 0.$$

So that the surface intersects the plane of xz in a *circle* and *ellipse*, whose equations are

$$r = b, \quad (a^2\cos.^2\alpha + c^2\sin.^2\alpha)\,r^2 = a^2c^2.$$

Now b, the radius of the circle, being intermediate between a and c, the semiaxes of the ellipse, it is obvious that the two curves must intersect in four points, or cusps, as represented in (fig. 1); and the angle which the radius-vector OP, drawn to the cusp, makes with the axis of x, is found by eliminating r between the two equations, by which means we obtain

$$\sin.\alpha = \pm\frac{a}{b}\sqrt{\frac{b^2-c^2}{a^2-c^2}}.$$

At each of the points thus determined there will be two tangents to the plane section; and consequently the ray OP, proceeding within the crystal to one of these points, might be supposed to be divided at emergence into two, whose directions are determined by those of the tangents.

Such seems to have been Fresnel's conception of this case. Professor Hamilton has shown, however, that there is a cusp at each of these points, not only in this particular section, but in every section of the wave surface passing through the line OP; or, in other words, that there is a *conoidal cusp* on the general wave-surface at the four points of intersection of the circle and ellipse; so that there must be an infinite number of tangent planes at each of these points, and consequently *a single ray*, such as OP, proceeding from a point within the crystal to one of these points, must be divided into an *infinite number of emergent rays*, constituting a conical surface.

It is evident further, that the circle and ellipse will have four common tangents, such as MN (fig. 1.) The planes passing through these tangents, and parallel to the third or mean axis, are parallel to the circular sections of the *surface of elasticity* of Fresnel's theory, or perpendicular to the optic axes. Fresnel seems to have concluded that these planes touched the wave-surface only in the two points just mentioned; and, consequently, that a single ray, incident upon a biaxal crystal in such a manner that one of the refracted rays should coincide with an optic axis, would be divided into two, determined by the points of contact. This result, if verified by experience, would place a remarkable distinction between the phenomena of uniaxal and biaxal crystals; but though the case was examined by M. BIOT, no corresponding appearances were observed.

Professor HAMILTON has shown that the four planes of which we have spoken touch the wave-surface, not in two points only, but in an infinite number of points, constituting each a small circle of contact, whose plane is parallel to one of the two circular sections of the surface of elasticity; and that, consequently, *a single ray* of common light, incident externally in the above-mentioned direction, should be divided within the crystal into an *infinite number of refracted rays*, constituting a conical surface.

Here. then, are two singular and unexpected consequences of the undulatory theory, not only unsupported by any facts hitherto observed, but even opposed to all the analogies derived from experience. If confirmed by experiment, they would furnish new and almost convincing proofs of the truth of that theory; and if disproved, on the other hand, it is evident that the theory must be abandoned or modified.

Being naturally anxious to submit the undulatory hypothesis to this delicate test, and to establish or disprove these new results of theory, Professor HAMILTON requested me to institute a series of experiments with that view. I accordingly applied myself to this interesting research with all the attention which the subject so well merited, and have fortunately succeeded in verifying both cases of conical refraction. The substance I employed in these experiments was *arragonite,* which is well known to be a biaxal crystal, whose axes are inclined at an angle of nearly 20°. I selected it partly on account of the magnitude of its biaxal energy, and partly also because the optical elements of this mineral have been determined, apparently with great care, by Professor RUDBERG; and therefore the results of theory could be applied to it at once without further examination. The specimen I used was one of considerable size and purity, procured for me by Mr. DOLLOND, and cut with its parallel faces perpendicular to the line bisecting the optic axes.

The first-mentioned species of conical refraction, it has been observed, takes place *in air*, when a ray of common light is transmitted within the crystal in the direction of the line joining two opposite cusps of the wave. If we suppose such a ray to pass in both directions out of the crystal, it is evident that it must emerge similarly at both surfaces; consequently, the rays which are transmitted along this line within the crystal, and form a diverging cone at emergence at the second surface, must be incident in a converging cone at the first. Having therefore nearly ascertained the required direction by means of the system of rings in polarized light, I placed a lens of short focus at its focal distance from the first surface, and in such a position that the central part of the pencil might have an incidence nearly corresponding to the cusp-ray within. Then looking through the crystal at the light of a lamp placed at a considerable distance, I observed in the expected direction a point more luminous than the space immediately about it, and surrounded by something resembling a stellar radiation. Fearing that this singular appearance might have arisen from some imperfection in the crystal, I transmitted the light in the same manner through several different parts of its substance, and always with the same result. The connexion of the phenomenon with the optic axis was proved by the system of rings which appeared in the same direction when the light was examined with a polarizing and analyzing plate.

This result is of some interest in itself, independently of its connexion with theory. It has been hitherto supposed that the only method of determining experimentally the direction of the optic axes, in most doubly refracting substances, consisted in observing the system of coloured rings, which appear around them when the incident and emergent light is polarized. Here, however, we find that common, or unpolarized light, undergoes such modifications in the neighbourhood of one of the optic axes, that the apparent direction of that axis may be at once determined, and with the aid of the simplest contrivance.*

But to examine the emergent cone, it was necessary to exclude the light which passed through the crystal in all but one direction. For this purpose, a plate of thin metal, having a minute aperture, was placed on the surface of the crystal next the eye, and the position of the aperture so adjusted, that the line connecting it with the luminous point on the first surface might be, as nearly as possible, in the direction of the cusp-ray. The exact adjustment to this direction was made by subsequent trial. The phenomenon which presented itself when this disposition was complete, was in the highest degree curious. There appeared at first a luminous circle, with a small dark space in the centre; and in this dark central space were two bright points, separated by a narrow and well-defined dark line. These appearances are represented in (figures *a* and *b*.) When the aperture in the plate was slightly shifted, the phenomena rapidly changed, assuming in succession the forms represented in (figs. *c*, *d*, *e*.) In the first stage of its change, the central dark space became greatly enlarged, and a double sector appeared in the centre. The circle was reduced to about a quadrant, and was separated by a dark interval from the sector just mentioned. This is represented in (fig. *c*.) The remote sector then disappeared, and the circular arch diminished, as in (fig. *d*);

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* This fact is here mentioned, rather as a matter of curiosity than as one likely to be of practical value in determining the optical elements of crystals. It is to be observed, moreover, that the direction thus determined is that of the normal to the circular section of the *ellipsoid* of FRESNEL's theory; while the rings (there is strong reason to believe) are related to the normals to the circular sections of the *surface of elasticity.*

and as the inclination of the internal ray to the cusp-ray was further increased, these two luminous portions merged gradually into the two pencils, into which a single ray is divided in the other parts of the crystal. This change is represented in (fig. *e*.)

Similar observations were made without the lens, by bringing the flame of the lamp near the first surface of the crystal, and forming the converging cone by covering that surface also with a thin metallic plate, perforated with a minute aperture. In this case the line connecting the two minute apertures was adjusted as before, and the phenomena were the same as in the former instance, the rays which passed along this line within the crystal forming a diverging cone at emergence.

In all these experiments the emergent rays were received directly by the eye placed close to the aperture on the second surface. It was obviously desirable, however, to receive them on a screen, and thus to observe the section of the cone at different distances from its summit. After some trials, I effected this with the sun's light, the light of a lamp being too weak for the purpose. The emergent cone being made to fall on a screen of roughened glass, I was enabled to observe its sections at various distances, and therefore with all the advantages of enlargement. The light was sufficiently bright, and the appearance distinct, when the diameter of the section was between one and two inches.

On examining the emergent cone with a tourmaline plate, I was surprised to observe that one radius only of the circular section* vanished in a given position of the axis of the tourmaline, and that the ray which disappeared ranged through 360° as the tourmaline plate was turned through 180°. Thus it appeared that all the rays of the cone are polarized in different planes.

On examining this curious phenomenon more attentively, I discovered the remarkable law, "*that the angle between the planes of polarization of any two rays of the cone is half the angle between the planes containing the rays themselves and the axis.*"

Having assured myself of the near truth of this law by experiment, I was naturally led to inquire how far it was in accordance with theory; and on examining Fresnel's theory with this view, I was gratified to find that it led to the very same result.

According to the known rule, the plane of polarization of any one ray of the emergent cone must bisect the angle contained by the planes passing through the corresponding normal to the front of the wave and the two optic axes. Now, it can be easily shown that the normals to the wave, at the cusp, surround one of the optic axes, and are inclined to it all round at small angles. For the tangent of the angle which the normals to the circle and ellipse in the plane of xz make with one another is

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* These sections are not mathematically circular, the line being, in fact, one of the fourth order.

$$\frac{\sqrt{a^2-b^2}\,\sqrt{b^2-c^2}}{a\,c};$$

and it can be easily shown that the tangent of the angle which the optic axis makes with the normal to the circle, or the cusp-ray, is

$$\frac{\sqrt{a^2-b^2}\,\sqrt{b^2-c^2}}{b^2+a\,c}$$

Now, this is about half the former, since $b^2=a\,c$, nearly; and consequently the optic axis nearly bisects the angle contained by the extreme normals in the plane of xz. Hence if A and B be the intersections of the two optic axes with the sphere whose centre is at the cusp, and N the intersection of one of the normals at that point with the same (fig. 2), the angle $N\,A\,C$ ranges through every magnitude between 0 and 360°, the arch $N\,A$ being all the time very small. Let the angle $N\,A\,C$ be denoted by α, and $N\,P\,C$ by ω, $N\,P$ being the arch bisecting the angle N; then in the triangle $A\,P\,N$, we have

$$\cos.\,\omega=\cos.\,A\,N.\ \sin.\,\alpha\,.\,\sin.\,\tfrac{1}{2}\,N+\cos.\,\alpha\,.\,\cos.\,\tfrac{1}{2}\,N;$$

or, since $A\,N$ is very small, and therefore $\cos.\,A\,N=1$, nearly,

$$\cos.\,\omega=\cos.\,(\alpha-\tfrac{1}{2}\,N),\text{ and }\omega=\alpha-\tfrac{1}{2}\,N,\text{ nearly.}$$

But, when any side of a spherical triangle is very small in comparison with the other two, the adjacent angles are together equal to 180° q. p. Consequently,

$$N=\alpha,\text{ and }\omega=\tfrac{1}{2}\,\alpha,\text{ nearly.}$$

From this it appears that the angle which the plane of polarization of any ray makes with the plane of the optic axes, is half the angle which the plane passing through the normal and the near axis makes with the same plane. But this latter angle, it may be easily shown, is very nearly the same as that which the plane passing through the emergent ray and the axis of the cone makes with the plane of the optic axes. Consequently, the angle which the plane of polarization of any ray of the emergent cone makes with the plane of the optic axes is half of that which the plane containing that ray and the axis of the cone forms with the same plane.

The general phenomena being observed, it remained to examine the magnitude and position of the emergent cone, and to compare the results with those furnished by theory. For this purpose I viewed the aperture in the second plate through a small telescope, which was moved in a plane nearly perpendicular to the axis of the emergent cone; and by noting the points at which the light failed, I obtained the magnitude of the section of the cone made by that plane. The distance of this section from the crystal being then measured, the angle of the cone was obtained from the trigonometrical tables; and was found to be very nearly 6°. I then placed the flame of a wax taper at the centre of the section, and removing the plate from the second surface of the crystal, found the direction of the ray reflected from the surface. A well defined mark was then placed on this line, at a considerable distance, and the angular distance between the centre of the flame and the mark measured by a sextant, whose centre was brought exactly to the place of the crystal. This angle was found to be 31° 56′; and consequently the angle of emergence corresponding to the central rays of the cone was 15° 58′.

Now to compare these results with those of theory.—It is a well-known principle of the theory of waves, that the direction of a ray incident upon, or emergent from, a crystal, and the normal to the front of the wave, are always in the same plane perpendicular to the surface of incidence or emergence; and the angles which these two lines make with the perpendicular to the surface, are connected by the known law of the sines; the index of refraction being the reciprocal of the normal velocity of the wave, or of the perpendicular upon the tangent plane. Now, at the cusp, there are an infinite number of normals to the wave, and consequently an infinite number of corresponding emergent rays. Of these the two rays in the plane of the optic axes form the greatest angle, and their directions are determined by those of the normals to the circle and ellipse, which constitute the section of the wave-surface in that plane. If then ρ' and ρ'' denote the angles of emergence of these rays, ι the angle which the normal to the circle, or cusp-ray, makes with the perpendicular to the surface, α the angle contained by the normals to the circle and ellipse, and p the perpendicular from the centre on the tangent to the ellipse at the cusp, we have

$$\sin.\ \rho' = \frac{1}{b} \sin.\ \iota\ , \quad \sin.\ \rho'' = \frac{1}{p} \sin.\ (\iota - \alpha)\ ;$$

In which

$$\frac{1}{p} = \frac{\sqrt{a^2+c^2-b^2}}{a\,c}, \quad \tan.\ \alpha = \frac{\sqrt{a^2-b^2}\ \sqrt{b^2-c^2}}{a\,c}\ .$$

Now in Arragonite, according to the determination of M. RUDBERG,

$$\frac{1}{a} = 1.5326\ , \quad \frac{1}{b} = 1.6863, \quad \frac{1}{c} = 1.6908\ ;$$

And substituting these values we find

$$\frac{1}{p} = 1{,}68708, \quad \alpha = 1^\circ.44'.48''.$$

These values being introduced in the first two equations, ρ' and ρ'' will be determined for any given surface of emergence. In this manner Professor HAMILTON has found that when $\iota = 0$, or the surface of emergence perpendicular to the cusp-ray, $\rho' = 0$, and $\rho' = 2^\circ.56'$-$51''$. And when $\iota = 9^\circ.\ 56'$-$27''$, or the surface perpendicular to the line bisecting the optic axes, $\rho' = 16^\circ\, 55'\, 27''$, and $\rho'' = 13^\circ\, 54'\, 49''$. Accordingly, the difference of these angles, $\rho' - \rho''$, which is the extreme angle of the emergent cone, is in the former case 2° 56' 51",* and in the latter 3° 0' 38". Also, half the sum of these angles, which is the angle of emergence corresponding to the axis of the cone, is 15° 25' 8".

Comparing these with the results of observation, it will be seen that they agree nearly with respect to the mean angle of emergence, the difference amounting only to 33'; whereas the angle of the cone determined by experiment is about double of that furnished by calculation.

I also measured the angle of the cone by tracing the outline of its section on a screen of roughened glass, when the sun's light was employed instead of that of a lamp. The mean diameter of this section being then accurately ascertained, and the distance of the screen from the aperture measured, the angle was given by the tables. Measurements taken in this manner gave for the value of the angle, 6° 24', 5° 56', 6° 22', respectively; and the mean of these is 6° 14', which, like the former measurement, differs very little from the double of the calculated angle.

The results of observation thus appeared to be at variance with those of theory in two important particulars. In the first place, the emergent rays appeared to form a *solid cone*, instead of a conical surface; and in the next, the magnitude of this cone was about double of the expected magnitude. Conceiving that these discrepancies might probably be owing to the rays which are inclined to the cusp-ray at small angles, and which pass by the edge of the aperture, I determined to ascertain the fact by trying the effects of apertures of various sizes.

I found accordingly that when the aperture was at all considerable, such as that formed by a large-sized pin, two concentric circles were seen to surround the axis, the interior of which had about double the brightness of the exterior annulus. And it was remarkable that the light of the interior circle was unpolarized, while that of the surrounding annulus was polarized according to the law already explained. When smaller apertures were used, the inner circle contracted, the breadth of the exterior annulus remaining nearly the same; until the former was finally reduced to a point in the centre of a fainter circle. When the aperture was still further diminished, a dark space sprung up in the centre, enlarging as the aperture decreased; until finally, with a very minute aperture, the breadth of this central space increased to about $\frac{3}{4}$ths of the entire diameter.

The phenomena exhibited in these cases assumed the forms represented in figures

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* It is easily shown that the sine of the angle of the cone, in this case, is generally expressed by the formula $\frac{\sqrt{a^2-b^2}\ \sqrt{b^2-c^2}}{a\,b\,c}$.

(*f*) and (*g*). (Fig. *h*) represents the appearance of the section when the line connecting the aperture with the luminous point on the first surface was slightly inclined to the cusp-ray.

It is easy to render an account of these various appearances. When the aperture *m n*, (fig. 3.) is at all considerable, the rays *c m*, *c n*, proceeding to its circumference from a point on the first surface, will be sensibly inclined to the cusp-ray, which we shall suppose to be the line *c o*, connecting the point on the first surface with the centre of the aperture. Consequently the interior refracted rays, *m q*, *n r*, as well as the exterior, *m p*, *n s*, will be inclined *outwards ;* and it is obvious that there will be a central bright space, limited by the lines *m q*, *n r*, each point of which will be illuminated by one interior and one exterior ray. The light in this space, therefore, will have double the intensity of that of the surrounding space ; and as the rays which combine to form it are polarized in planes at right-angles to one another, *the resulting light will be unpolarized.* When the aperture is diminished, the inclination of the rays *m q*, *n r*, to one another is lessened, until finally they are reduced to *parallelism,* and the central bright space contracts to a point. This is represented in (fig. 4.) When the aperture is still further diminished, the rays *m q*, *n r*, become inclined *inwards,* and cross (fig. 5.) It is obvious that beyond the point of intersection there will be a dark space illumined by no ray whatever ; and as in the surrounding annulus there is no meeting of rays oppositely polarized, *the whole of the light will be polarized,* and according to the law already explained. With a yet diminished aperture, the rays *m q*, *n r*, approach to parallelism with the exterior rays, *n s*, *m p ;* and the central dark space enlarges, and approaches to equality with the outer and limiting cone. Thus the annulus of light in any section is diminished indefinitely in breadth, and the cone approaches to a mathematical surface.

Now if we assume that the divergence of the two refracted rays in this plane, corresponding severally to the rays *c m*, *c o*, *c n*, is the same, as must be nearly the case, it will follow that the angle of the true cone, which would arise from the single ray *c o*, is half the sum of the angles of the exterior and interior surfaces of the conical annulus ; and that when a bright circle appears in the centre, as is the case with a considerable aperture, the dark space must be considered as negative, and the true angle is half the difference of the observed angles.

From this it follows that when the central bright space is reduced to a point, the true angle is just half the observed. Now this was very nearly the case in the experiments from which the measures were taken ; consequently the corrected angle, deduced from these measures, coincides very nearly with that assigned by theory.

Two other measurements, taken since with a more direct reference to this correction, were as follows :—

1. Distance of screen from the aperture on the second surface of the crystal = 19.3 half inches. Mean diameter of section of exterior cone = 1.27. Mean diameter of interior = 0.55. Corrected angle of cone thence computed = 2° 44′.

2. Distance of screen = 11.9. Mean diameter of section of exterior cone = 0.93. Mean diameter of interior = 0.41. Computed angle of cone = 3° 14′.

The mean of these two measurements is 2° 59′.

Inasmuch as the cusp-ray, within the crystal, corresponds to a cone of rays without, it is evident that there must be a *converging* cone incident on the first surface, equal to that which diverges from the second. With a view to determine its magnitude, I placed a kind of rough micrometer, consisting of two moveable metallic plates, immediately before the lens; and closed the plates until, on looking through the aperture on the second surface, I could see them touching the circumference of the annular section. The diameters of the interior and exterior circumferences of this section, at the distance of the lens, being thus ascertained, and the focal length of the lens measured, the corrected angle of the cone was found. The mean of three measurements taken in this manner gave for this angle 3° 47′. But the methods by which this last result was obtained, do not seem susceptible of much accuracy.

Before I conclude this part of the subject, I may observe that an interesting variation in the phenomena is obtained by substituting a narrow linear aperture for the small circular one, in the plate which covers the first surface of the crystal—that surface being close to the lamp. The linear aperture is to be so fixed, that the plane passing through it and the aperture in the plate next the eye, shall be the plane of the optic axes. In this case, according to the received theory, all the rays transmitted through the two apertures should be refracted doubly in the plane of the optic axes, so that no part of the line should appear enlarged in breadth on looking through the second aperture; whereas, according to Professor HAMILTON's beautiful conclusion from the same theory, the cusp-ray should be refracted in every possible azimuth. I found accordingly that the luminous line was undilated, except in the direction corresponding to that of the cusp-ray; and that in the neighbourhood of this direction its boundaries were no longer rectilinear, but swelled out in the form of an oval curve (fig. *i.*)

When a very minute aperture was used on the surface next the eye, in this experiment, the phenomenon was rendered much more remarkable. The swelling curves in this case were separated by a considerable dark interval, and the luminous line was prolonged into this dark space, terminating abruptly near its centre. This appearance is represented in (fig. *k.*) When the plate next the eye was slightly shifted, so that the plane passing through the two apertures no longer coincided accurately with the plane of the optic axes, the curves rapidly changed, preserving, however, in all cases, the form of the *conchoid*, whose pole was the projection of the axis of the emergent cone, and asymptot the line on the first surface—(figs. *l*, *m*.) It is easy to show that these results are in accordance with theory.

The second kind of *conical refraction*, whose existence has been anticipated by Professor HAMILTON, depends (it will be remembered) on the mathematical fact, that the wave-surface is touched in an infinite number of points, constituting a small *circle of contact*, by a single plane parallel to one of the circular sections of the surface of elasticity. It takes place when a *single external ray* falls upon a biaxal crystal in such a manner, that one refracted ray may coincide with an optic axis. When this is the case, there will be a *cone of rays* within the crystal, determined by lines connecting the centre of the wave with the points of the periphery of the circle of contact. The angle of this cone is equal to

$$\text{tang.}^{-1}\ \frac{\sqrt{a^2-b^2}\ \sqrt{b^2-c^2}}{b^2};$$

and its numerical value in the case of Arragonite is 1° 55′, assuming the values of the three indices as determined for the ray E by Professor RUDBERG; (see page 151.)

As the rays constituting this cone will be refracted at emergence in a direction parallel to the incident ray, they will form a small cylinder of rays in air. This cylinder, it will be seen, is in all cases extremely small; for the diameter of its section made by the surface of emergence subtends an angle of 1° 55′ only, at a distance equal to the thickness of the crystal. Hence the experiments required to detect its existence and measure its magnitude, demand more care and precision than those already described. The incident light was that of a lamp placed at some distance; and in order to reduce as much as possible the breadth of the incident beam, it was constrained to pass through two small apertures, the first of which was in a screen placed near the flame, and the second perforated in a thin metallic plate adjoining to the first surface of the crystal. Under ordinary circumstances, it is obvious, the incident ray will be divided into two within the crystal, and these will emerge parallel from the second surface. I was able to distinguish these two rays by the aid of a lens; and turning the crystal slowly, so as to vary the incidence gradually, I at length observed that there was a position in which the two rays changed their relative places rapidly, on any slight change of incidence, and appeared at times to revolve round one another, as the incidence was altered. Being convinced that the ray was now near the critical incidence, I changed the position of the crystal, with respect to the incident ray, very slowly; and after much care in the adjustment, I at last saw the two rays spread into a continuous circle, whose diameter was apparently equal to their former interval.

This phenomenon was exceedingly striking. It looked like a small ring of gold viewed upon a dark ground; and the sudden and almost magical change of the appearance from two luminous points to a perfect luminous ring, contributed not a little to enhance the interest.

The emergent light, in this experiment, being too faint to be reflected from a screen, I repeated the experiment with the sun's light, and received the emergent cylinder upon a small piece of silver-paper. I could detect no sensible difference in the magnitude of the circular sections at different distances from the crystal.

When the adjustment was perfect, the light of the entire annulus was white, and of equal intensity throughout. But when there was a very slight deviation from the exact position, two opposite quadrants of the circle appeared more faint than the other two, and the two pairs were of complementary colours.* The light of the circle was polarized, according to the law which I had before observed in the other case of conical refract In this instance, however, the law was anticipated from theory by Professor HAMILTON.

I measured the angle of incidence by a method similar to that already employed for the emergent ray in the former case; and found it to be 15° 40′. This determination is, for many reasons, capable of much greater accuracy than the other; and was probably, in this instance, very near the truth.

In order to compare it with the result of theory, it is to be observed that the optic axis is a normal to the wave-surface, and therefore the corresponding incident ray will be given by the ordinary law of the sines, the index of refraction being the *mean index* of the crystal. Now the angle which the normal to the circular section of the surface of elasticity, or the optic axis, makes with the axis of x, or the perpendicular to the surface, is equal to $\text{tang.}^{-1}\sqrt{\frac{b^2-c^2}{a^2-b^2}}$; and its numerical value in the case of Arragonite, is 9° 1′. We have then

$$\sin.\,\iota = 1.6863,\ \sin.\,(9^\circ\ 1');$$

from which we find $\iota = 15^\circ\ 19'$. The difference between this and the observed angle is 21′.

In order to measure the angle of the cone, I was compelled to employ a method somewhat indirect, but (I think) susceptible of considerable accuracy. As the aperture on the first surface of the crystal must have some physical magnitude, it is obvious that instead of a cone of mathematical rays within the crystal, there will be in all cases a cone of cylindrical pencils, overlapping one another near the point of divergence; and that the diameter of these pencils will be equal to the diameter of the aperture. Now I tried a number of apertures, until I found one with which these cylindrical pencils just separated at the second surface of the crystal. It is evident that, in this case, the interval between the axes of the cylinders at the surface of emergence, is just equal to the diameter of one of them, or to the diameter of the aperture. I had then only to measure the aperture itself. This was effected by the aid of a micrometer divided to the 1-500th of an inch, placed along with the aperture before a compound microscope; and it was found to be .016 of an inch. This therefore was the diameter of the oblique section of the cone made by the surface of emergence; and the diameter of the circular section at the same distance was .016 cos. 9°, since the axis of the cone makes an angle of 9° with the normal to the faces of the crystal. The perpendicular thickness of the crystal was .49 of an inch; and therefore the thickness estimated in the direction of the axis of the cone was $\frac{.49}{\cos. 9^\circ}$. From these data the angle of the cone was calculated by the tables, and found to be 1° 50′; a result which differs from the theoretical angle by 5′ only.

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* This part of the phenomenon appears to be explained by the non-coincidence of the optic axes for the rays of different colours.